Phase semantics of linear logic applied to the focalisation property

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B^{\perp}, B}\,ax}{\vdash B^{\perp}, B \oplus D}\,\oplus_1 \qquad \dfrac{\dfrac{}{\vdash C^{\perp}, C}\,ax}{\vdash C^{\perp}, C \oplus E}\,\oplus_1}{\vdash B^{\perp} \otimes C^{\perp}, B \oplus D, C \oplus E}\,\otimes}{\vdash A \oplus B^{\perp} \otimes C^{\perp}, B \oplus D, C \oplus E}\,\oplus_2$$

1 Introduction

Linear logic, introduced by Jean-Yves Girard in the late 1980s [Gir87], is a logic that stems from the continuation of the path from classical to intuitionistic logic, and has proven to have many fruitful applications, from functional programming to quantum mechanics. The most common interpretation of linear logic is in term of *resources*: where classical logic allows one to use each hypothesis as many times as one wants (including not at all) by using the weakening and contraction rules, linear logic gets rid of these rules, hence requiring that each premise be used *exactly* once, as in e.g. a chemical reaction.

During this internship, I have been investigating the phase semantics of linear logic, a simple semantics of *provability*, and trying to extract information on *proofs* from such semantics, in particular in relationship to the focalisation property which allows a proof search procedure to narrow down its search space in certain situations.

We will start with a basic review of propositional linear logic in section 2, then we will introduce phase semantics and their main completeness results in section 3, and finally we will explore the application of phase semantics to the focalisation property in section 4.

2 Linear logic

Linear logic arises from the removal of the weakening and contraction rules from classical logic. This prohibition splits the usual connectives of classical logic into *multiplicative* and *additive* variants:

classical	LL (multiplicative)	LL (additive)
$\wedge$	$\otimes$	$\&$
$\vee$	$⅋$	$\oplus$
$\top$	1	$\top$
$\bot$	$\bot$	0

where $\otimes$, $\oplus$, $\&$ and $⅋$ are associative and commutative, and 1, 0, $\top$ and $\bot$ are their respective units (and nullary versions).

In order to keep the expressive power of classical logic, we reintroduce weakening and contraction in a *controlled* manner under the connectives ! and ?. In the resource interpretation, $!A$ can be understood as an infinite source of A that can be used or discarded as one wishes.

There is an elegant symmetry in linear logic, embodied by the involutive *linear negation* $\cdot^\perp$, which is defined inductively on formulas using De Morgan laws:

$$
\begin{aligned}
(A \otimes B)^\perp &= A^\perp ⅋ B^\perp & \qquad (A ⅋ B)^\perp &= A^\perp \otimes B^\perp \\
(A \oplus B)^\perp &= A^\perp \& B^\perp & (A \& B)^\perp &= A^\perp \oplus B^\perp \\
1^\perp &= \bot & \bot^\perp &= 1 \\
0^\perp &= \top & \top^\perp &= 0 \\
(!A)^\perp &= ?A & (?A)^\perp &= !A \\
(X)^\perp &= X^\perp & (X^\perp)^\perp &= X
\end{aligned}
$$

We now present the one-sided sequent calculus for linear logic, LL. A context (noted Γ, Δ, $\ldots$) is a multiset of formulas, and a sequent has the general form $\Gamma \vdash \Delta$. However, such a sequent is equivalent to $\vdash \Gamma^\perp, \Delta$, so it is enough to consider sequents with only formulas on the right.

The inference rules for LL are as follows:

$$
\frac{}{\vdash A^\perp, A}\ ax \qquad \frac{\vdash \Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ cut
$$

$$
\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A ⅋ B}\ ⅋ \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \bot
$$

$$
\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{}{\vdash 1}\ 1
$$

$$
\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \& B}\ \& \qquad \frac{}{\vdash \Gamma, \top}\ \top
$$

$$
\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \oplus_1 \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus_2
$$

$$
\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ! \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ ?d \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ ?w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ ?c
$$

Linear implication, noted $A \multimap B$, is defined as $A^\perp \parr B$.

Remarkably, $\otimes$ and $\oplus$ distribute over each other; dually, $\parr$ and $\&$ distribute over each other; finally, we have $!(A \& B) \equiv\ !A \otimes !B$.

3 Phase semantics

Linear logic has two main semantics: coherence spaces, which are a semantics of *proofs*, and phase semantics, a simpler semantics of *provability* which we will focus on.

Definition 3.1. *A **phase space** consists of a commutative monoid M of* phases *(written multiplicatively) and a subset $\perp \subseteq M$ of* antiphases.

The linear negation of a set of phases $X \subseteq M$ is defined as $X^\perp = \{m \in M \mid \forall x \in X, m \cdot x \in \perp\}$. For any $X, Y \subseteq M$, we have the following easy results:

- $X \subseteq X^{\perp\perp}$
- $X \subseteq Y \implies Y^\perp \subseteq X^\perp$
- $X^\perp = X^{\perp\perp\perp}$
- $(X \cup Y)^\perp = X^\perp \cap Y^\perp$

A *fact* is a set of phases X such that $X = X^{\perp\perp}$. Equivalently, a fact is a set of the form $Y^\perp$ for $Y \subseteq M$.

We define the following operators and constants on $\mathcal{P}(M)$:

$$X \otimes Y = (X \cdot Y)^{\perp\perp} \qquad X \parr Y = (X^\perp \cdot Y^\perp)^\perp$$
$$X \oplus Y = (X \cup Y)^{\perp\perp} \qquad X \& Y = X \cap Y$$
$$\mathbf{1} = \perp^\perp$$
$$\mathbf{0} = \top^\perp \qquad \top = M$$
$$I = \{m \in M \mid m \cdot m = m\} \text{ (the set of } \textit{idempotents}\text{)}$$
$$!X = (X \cap \mathbf{1} \cap I)^{\perp\perp} \qquad ?X = (X^\perp \cap \mathbf{1} \cap I)^\perp$$

Let Φ be an n-ary monotonous operator on $\mathcal{P}(M)$. Φ is

– **negative** if it maps facts to facts, i.e. $\Phi(X_1^{\perp\perp}, \ldots, X_n^{\perp\perp})^{\perp\perp} = \Phi(X_1^{\perp\perp}, \ldots, X_n^{\perp\perp})$;

+ **positive** if it verifies $\Phi(X_1^{\perp\perp}, \ldots, X_n^{\perp\perp}) \subseteq \Phi(X_1, \ldots, X_n)^{\perp\perp}$ (and hence, by monotonicity, $\Phi(X_1^{\perp\perp}, \ldots, X_n^{\perp\perp})^{\perp\perp} = \Phi(X_1, \ldots, X_n)^{\perp\perp}$).

In the nullary case, every set of phases is positive and the negative sets of phases are exactly facts.

Note that $\cdot$ and $\cup$ (and their n-ary versions) are positive operators, while $\parr$ and $\&$ are negative operators [Gir99, appendix F], and that both properties are stable under composition.

Definition 3.2. *A **phase model** is a phase space $(M, \perp)$ together with a fact $[\![X]\!]$ for every atomic formula X. The interpretation $[\![A]\!]$ of a formula A is defined by induction using the operators above, and the interpretation of a context $\Gamma = A_1, \ldots, A_n$ is defined as $[\![\Gamma]\!] = [\![A_1]\!] \parr \cdots \parr [\![A_n]\!]$. One can easily check that $[\![A]\!]$, $[\![\Gamma]\!]$ are facts.*

A formula A is said to be *valid* in a given phase model if $1 \in [\![A]\!]$. More generally, a sequent $\vdash \Gamma$ is valid if $1 \in [\![\Gamma]\!]$.

Let us now state core results of phase semantics:

Theorem 3.1 (Soundness)**.** *If a sequent is provable in LL, then it is valid in every phase model.*

Proof. By induction on a proof of LL; see [Oka98, theorem 1]. □

Theorem 3.2 (Completeness)**.** *If a sequent is valid in every phase model, then it is provable in LL.*

This statement can actually be made stronger:

Theorem 3.3 (Cut-free completeness)**.** *If a sequent is valid in every phase model, then it has a cut-free proof in LL.*

Proof. The proof proceeds by defining a **syntactic phase model** $(M, \perp)$ where M is the free commutative monoid over formulas of LL with $?A$ and $?A, ?A$ identified, $\perp = \{\Gamma \in M \mid \vdash \Gamma \text{ is provable without the } \textit{cut} \text{ rule}\}$, and $[\![X]\!] = \{X\}^{\perp}$. It then shows that $[\![A]\!] \subseteq \{A\}^{\perp}$ by induction on formulas, from which the result follows; see [Oka98, theorem 3] for details. □

Combining the cut-free completeness theorem with the soundness theorem, we get:

Theorem 3.4 (Cut elimination)**.** *If a sequent is provable in LL, then it has a cut-free proof.*

4 The focalisation property

Proof search is the problem of finding a proof of a given sequent. It can be expressed recursively, starting from a root sequent and working its way up towards the leaves (e.g. axiom rules). At each step, the procedure must choose a formula in the current sequent, a rule to obtain that formula, and possibly the prerequisites for that rule. In fact, we can observe that the only connectives whose introduction rule requires making a choice are $\otimes$ and $\oplus$: the $\otimes$ rule requires choosing a way to split the context into two, while the $\oplus$ connective requires choosing between the $\oplus_1$ and $\oplus_2$ rules.

This suggests another partition of the connectives into two *polarities*, the *positive* connectives ($\otimes$, $\oplus$, 1, 0, $!$ and positive atoms) and the *negative* connectives ($⅋$, $\&$, $\perp$, $\top$, $?$ and negative atoms). Notice that the remarkable distributivities of section 2 only occur between connectives of the same polarities, and that linear negation flips the polarity of a formula (which is defined as the polarity of its main connective).

A crucial property of the proof theory of linear logic is the *focalisation* (or *focusing*) property, discovered by Jean-Marc Andreoli [And92], which allows the search space to be reduced by "focusing" on certain connectives.

The property has two sides:

− negative connectives are *reversible*: in a sequent with negative formulas, we can always start by applying the introduction rules for the negative connectives without risk;

\+ positive connectives can be grouped into a maximal cluster and handled all at once. For example, given the formula $A \oplus (B \otimes C)$, if we decide to decompose it using the $\oplus_2$ rule, then we can immediately decompose $B \otimes C$ using the $\otimes$ rule without needing to backtrack.

The focalisation property already has several syntactic proofs [And92; SM07; Lau04]; the main contribution of this internship is a *semantic* proof of the completeness of focused proofs using phase semantics.

Laurent's proof [Lau04] proceeds in two steps: it first embeds proofs of LL into proofs of $\mathsf{LL}_{\mathrm{foc}}$, a restricted variant of LL that enforces a *weak* focalisation property (the core of the proof, according to Laurent), then it embeds cut-free proofs of $\mathsf{LL}_{\mathrm{foc}}$ into proofs of $\mathsf{LL}_{\mathrm{Foc}}$ (note the capital), an even more restricted system that enforces the full focalisation property.

We prove that phase semantics are complete with respect to the cut-free $\mathsf{LL}_{\mathrm{foc}}$ system in a way similar to the proof of theorem 3.3. A sequent of $\mathsf{LL}_{\mathrm{foc}}$ has the shape $\vdash \Gamma; \Pi$, where Γ is a multiset of formulas and Π contains at most one positive formula. To simplify the notation, we let $\vdash \Gamma; N$ mean $\vdash \Gamma, N;$ when N is a negative formula. The rules of the cut-free $\mathsf{LL}_{\mathrm{foc}}$ system with expanded axioms are as follows:

$$\frac{}{\vdash X^\perp; X}\, ax \qquad \frac{\vdash \Gamma; P}{\vdash \Gamma, P}\, foc$$

$$\frac{\vdash \Gamma, A, B; \Pi}{\vdash \Gamma, A \parr B; \Pi}\, \parr \qquad \frac{\vdash \Gamma; \Pi}{\vdash \Gamma, \perp; \Pi}\, \perp$$

$$\frac{\vdash \Gamma; A \qquad \vdash \Delta; B}{\vdash \Gamma, \Delta; A \otimes B}\, \otimes \qquad \frac{}{\vdash ; 1}\, 1$$

$$\frac{\vdash \Gamma, A; \Pi \qquad \vdash \Gamma, B; \Pi}{\vdash \Gamma, A \,\&\, B; \Pi}\, \& \qquad \frac{}{\vdash \Gamma, \top; \Pi}\, \top$$

$$\frac{\vdash \Gamma; A}{\vdash \Gamma; A \oplus B}\, \oplus_1 \qquad \frac{\vdash \Gamma; B}{\vdash \Gamma; A \oplus B}\, \oplus_2$$

$$\frac{\vdash ?\Gamma, A;}{\vdash ?\Gamma; !A}\, ! \qquad \frac{\vdash \Gamma; A}{\vdash \Gamma, ?A;}\, ?d \qquad \frac{\vdash \Gamma; \Pi}{\vdash \Gamma, ?A; \Pi}\, ?w \qquad \frac{\vdash \Gamma, ?A, ?A; \Pi}{\vdash \Gamma, ?A; \Pi}\, ?c$$

Definition 4.1. *We define the* ***focalised syntactic phase model*** *as $(M, \perp)$ where M is the free commutative monoid over formulas of LL with $?A$ and $?A, ?A$ identified, $\perp = \{\Gamma \in M \mid \vdash \Gamma;\}$, and $[\![X]\!] = \{X^\perp\}^{\perp\perp}$ for positive atoms X.*

Let $\mathrm{Foc}(A) = \{\Gamma \in M \mid \vdash \Gamma; A\}$. Clearly $\mathrm{Foc}(A) \subseteq \{A\}^\perp$ by the *foc* rule, and in particular $\mathrm{Foc}(N) = \{N\}^\perp$ for N negative.

Note that provability is compatible with our identification of $?A$ and $?A, ?A$ thanks to the $?w$ and $?c$ rules, so that $\perp$ and $\mathrm{Foc}(A)$ are well-defined. Also note that $I = \{?\Gamma \mid \Gamma \in M\} \subseteq \mathbf{1}$ because of the $?w$ rule, so that $[\![!A]\!] = ([\![A]\!] \cap I)^{\perp\perp}$.

We use the decomposition of exponential connectives alluded to in [Lau04, section 4.1]:

$$!A = {\downarrow}{\sharp}A \qquad\qquad ?A = {\uparrow}{\flat}A$$

where $\downarrow, \flat$ are positive and $\sharp, \uparrow$ are negative. In fact, we only need to consider formulas of the forms A and $\sharp A$, where A is a formula of LL. Let us extend our definitions to this connective with $[\![\sharp A]\!] = [\![!A]\!]$ and $\mathrm{Foc}(\sharp A) = \{A\}^\perp \cap I$.

For a formula A, let $|A|$ denote the number of main negative subformulas in A (where $\sharp B$ is the main negative subformula in $!B$).

Let Ψ_A be an $|A|$-ary positive operator on $\mathcal{P}(M)$ defined by induction as follows:

- $\Psi_N(N_1) = N_1$ if N is negative

- $\Psi_X() = \{X^\perp\}$
- $\Psi_{B\otimes C}(B_1, \ldots, B_{|B|}, C_1, \ldots, C_{|C|}) = \Psi_B(B_1, \ldots, B_{|B|}) \cdot \Psi_C(C_1, \ldots, C_{|C|})$
- $\Psi_{B\oplus C}(B_1, \ldots, B_{|B|}, C_1, \ldots, C_{|C|}) = \Psi_B(B_1, \ldots, B_{|B|}) \cup \Psi_C(C_1, \ldots, C_{|C|})$
- $\Psi_1() = \{\emptyset\}$
- $\Psi_0() = \emptyset$
- $\Psi_{!B}(B_1) = B_1$

Lemma 4.1. *For any formula A with main negative subformulas $A_1, \ldots, A_{|A|}$,*

$$[\![A]\!] = \Psi_A([\![A_1]\!], \ldots, [\![A_{|A|}]\!])^{\perp\perp}$$

Proof. By induction:

- If A is negative, then we have $[\![A]\!] = [\![A]\!]^{\perp\perp}$ because $[\![A]\!]$ is a fact.
- If $A = X$, then $[\![X]\!] = \{X^\perp\}^{\perp\perp} = \Psi_X()^{\perp\perp}$.
- If $A = B \otimes C$, then

$$\begin{aligned}
[\![B \otimes C]\!] &= ([\![B]\!] \cdot [\![C]\!])^{\perp\perp} \\
&= (\Psi_B([\![B_1]\!], \ldots, [\![B_{|B|}]\!])^{\perp\perp} \cdot \Psi_C([\![C_1]\!], \ldots, [\![C_{|C|}]\!])^{\perp\perp})^{\perp\perp} && \text{by the induction hypothesis} \\
&= (\Psi_B([\![B_1]\!], \ldots, [\![B_{|B|}]\!]) \cdot \Psi_C([\![C_1]\!], \ldots, [\![C_{|C|}]\!]))^{\perp\perp} && \text{by positivity} \\
&= \Psi_{B\otimes C}([\![B_1]\!], \ldots, [\![B_{|B|}]\!], [\![C_1]\!], \ldots, [\![C_{|C|}]\!])^{\perp\perp}
\end{aligned}$$

- If $A = B \oplus C$, then

$$\begin{aligned}
[\![B \oplus C]\!] &= ([\![B]\!] \cup [\![C]\!])^{\perp\perp} \\
&= (\Psi_B([\![B_1]\!], \ldots, [\![B_{|B|}]\!])^{\perp\perp} \cup \Psi_C([\![C_1]\!], \ldots, [\![C_{|C|}]\!])^{\perp\perp})^{\perp\perp} && \text{by the induction hypothesis} \\
&= (\Psi_B([\![B_1]\!], \ldots, [\![B_{|B|}]\!]) \cup \Psi_C([\![C_1]\!], \ldots, [\![C_{|C|}]\!]))^{\perp\perp} && \text{by positivity} \\
&= \Psi_{B\oplus C}([\![B_1]\!], \ldots, [\![B_{|B|}]\!], [\![C_1]\!], \ldots, [\![C_{|C|}]\!])^{\perp\perp}
\end{aligned}$$

- If $A = 1$ then $[\![1]\!] = \{\emptyset\}^{\perp\perp}$ by definition.
- If $A = 0$ then $[\![0]\!] = \emptyset^{\perp\perp}$ by definition.
- If $A = {!B}$ then $[\![!B]\!] = [\![\sharp B]\!] = \Psi_{!B}([\![\sharp B]\!])^{\perp\perp}$.

$\square$

Lemma 4.2. *For any formula A with main negative subformulas $A_1, \ldots, A_{|A|}$,*

$$\Psi_A(\mathrm{Foc}(A_1), \ldots, \mathrm{Foc}(A_{|A|})) \subseteq \mathrm{Foc}(A)$$

Proof. By induction:

- If A is negative, $\Psi_A(\mathrm{Foc}(A)) = \mathrm{Foc}(A)$ by definition.
- If $A = X$, then $\Psi_X() = \{X^\perp\} \subseteq \mathrm{Foc}(X)$ by the *ax* rule.
- If $A = B \otimes C$, then

$$\begin{aligned}
&\Psi_A(\mathrm{Foc}(A_1), \ldots, \mathrm{Foc}(A_{|A|})) \\
&= \Psi_B(\mathrm{Foc}(B_1), \ldots, \mathrm{Foc}(B_{|B|})) \cdot \Psi_C(\mathrm{Foc}(C_1), \ldots, \mathrm{Foc}(C_{|C|})) \\
&\subseteq \mathrm{Foc}(B) \cdot \mathrm{Foc}(C) && \text{by the induction hypothesis} \\
&\subseteq \mathrm{Foc}(B \otimes C) && \text{by the } \otimes \text{ rule}
\end{aligned}$$

- If $A = B \oplus C$, then

$$\begin{aligned}
&\Psi_A(\mathrm{Foc}(A_1), \ldots, \mathrm{Foc}(A_{|A|})) \\
&= \Psi_B(\mathrm{Foc}(B_1), \ldots, \mathrm{Foc}(B_{|B|})) \cup \Psi_C(\mathrm{Foc}(C_1), \ldots, \mathrm{Foc}(C_{|C|})) \\
&\subseteq \mathrm{Foc}(B) \cup \mathrm{Foc}(C) && \text{by the induction hypothesis} \\
&\subseteq \mathrm{Foc}(B \oplus C) && \text{by the } \oplus_1 \text{ and } \oplus_2 \text{ rules}
\end{aligned}$$

- If $A = 1$, then $\Psi_1() = \{\emptyset\} \subseteq \mathrm{Foc}(1)$ by the 1 rule.
- If $A = 0$, clearly $\Psi_0() = \emptyset \subseteq \mathrm{Foc}(0)$.
- If $A = !B$, then

$$\begin{aligned}
\Psi_{!B}(\mathrm{Foc}(\sharp B)) &= \{B\}^\perp \cap I \\
&\subseteq \mathrm{Foc}(!B) && \text{by the ! rule}
\end{aligned}$$

□

Lemma 4.3. *For any formula A, $[\![A]\!] \subseteq \mathrm{Foc}(A)^{\perp\perp}$.*

Proof. By induction:

- If A is a positive formula with main negative subformulas $A_1, \ldots, A_{|A|}$, then

$$\begin{aligned}
[\![A]\!] &= \Psi_A([\![A_1]\!], \ldots, [\![A_{|A|}]\!])^{\perp\perp} && \text{by lemma 4.1} \\
&\subseteq \Psi_A(\mathrm{Foc}(A_1)^{\perp\perp}, \ldots, \mathrm{Foc}(A_{|A|})^{\perp\perp})^{\perp\perp} && \text{by the induction hypothesis} \\
&= \Psi_A(\mathrm{Foc}(A_1), \ldots, \mathrm{Foc}(A_{|A|}))^{\perp\perp} && \text{by positivity} \\
&\subseteq \mathrm{Foc}(A)^{\perp\perp} && \text{by lemma 4.2}
\end{aligned}$$

- If $A = \sharp B$, then by the induction hypothesis $[\![\sharp B]\!] = [\![!B]\!] = ([\![B]\!] \cap I)^{\perp\perp} \subseteq (\mathrm{Foc}(B)^{\perp\perp} \cap I)^{\perp\perp} \subseteq (\{B\}^\perp \cap I)^{\perp\perp} = \mathrm{Foc}(\sharp B)^{\perp\perp}$.

  Otherwise, it is enough to prove $[\![A]\!] \subseteq \{A\}^\perp$.
- If $A = X^\perp$, then $[\![X^\perp]\!] = [\![X]\!]^\perp = \{X^\perp\}^{\perp\perp\perp} = \{X^\perp\}^\perp$.

- If $A = B \,\&\, C$, we have $[\![B \,\&\, C]\!] = [\![B]\!] \cap [\![C]\!] \subseteq \{B\}^\perp \cap \{C\}^\perp$ by the induction hypothesis; moreover,

$$\frac{\vdash \Gamma, B; \qquad \vdash \Gamma, C;}{\vdash \Gamma, B \,\&\, C;}\ \&$$

hence $\{B\}^\perp \cap \{C\}^\perp \subseteq \{B \,\&\, C\}^\perp$, from which the result follows.

- If $A = B \parr C$, let $\Gamma \in [\![B \parr C]\!] = ([\![B]\!]^\perp \cdot [\![C]\!]^\perp)^\perp$. By the induction hypothesis, $[\![B]\!] \subseteq \{B\}^\perp$, hence $B \in \{B\}^{\perp\perp} \subseteq [\![B]\!]^\perp$, and similarly $C \in [\![C]\!]^\perp$, therefore $\vdash \Gamma, B, C;$. Moreover,

$$\frac{\vdash \Gamma, B, C;}{\vdash \Gamma, B \parr C;}\ \parr$$

hence $\Gamma \in \{B \parr C\}^\perp$, from which the result follows.

- If $A = \top$, we have $[\![\top]\!] = M = \{\top\}^\perp$ by the $\top$ rule.

- If $A = \bot$, we have $[\![\bot]\!] = \bot \subseteq \{\bot\}^\perp$ by the $\bot$ rule.

- If $A = ?B$, then $[\![?B]\!] = ([\![B]\!]^\perp \cap I)^\perp$. By the induction hypothesis, $[\![B]\!] \subseteq \mathrm{Foc}(B)^{\perp\perp}$, hence $([\![B]\!]^\perp \cap I)^\perp \subseteq (\mathrm{Foc}(B)^\perp \cap I)^\perp$. Moreover, $?B \in \mathrm{Foc}(B)^\perp \cap I$ because of the $?d$ rule, therefore $(\mathrm{Foc}(B)^\perp \cap I)^\perp \subseteq \{?B\}^\perp$, from which the result follows.

□

Corollary 4.3.1. *For any context $\Gamma = A_1, \ldots, A_n$, $[\![\Gamma]\!] \subseteq \{\Gamma\}^\perp$.*

Proof. By lemma 4.3, we have $[\![A_i]\!] \subseteq \mathrm{Foc}(A_i)^{\perp\perp} \subseteq \{A_i\}^\perp$ for all $1 \le i \le n$, hence $\{A_i\} \subseteq \{A_i\}^{\perp\perp} \subseteq [\![A_i]\!]^\perp$, therefore $\{\Gamma\} = \{A_1\} \cdots \{A_n\} \subseteq [\![A_1]\!]^\perp \cdots [\![A_n]\!]^\perp$.

Thus, $[\![\Gamma]\!] = [\![A_1]\!] \parr \cdots \parr [\![A_n]\!] = ([\![A_1]\!]^\perp \cdots [\![A_n]\!]^\perp)^\perp \subseteq \{\Gamma\}^\perp$. □

Theorem 4.4 (Cut-free completeness in $\mathsf{LL}_{\mathrm{foc}}$)**.** *If a sequent $\vdash \Gamma$ of LL is valid in all phase models, then $\vdash \Gamma;$ has a cut-free proof in $\mathsf{LL}_{\mathrm{foc}}$.*

Proof. We have $\emptyset \in [\![\Gamma]\!]$, hence $\emptyset \in \{\Gamma\}^\perp$ by corollary 4.3.1, therefore there is a cut-free proof of $\vdash \Gamma;$ in $\mathsf{LL}_{\mathrm{foc}}$. □

Combining this with the soundness theorem and [Lau04, proposition 1], we get:

Theorem 4.5 (Weak focalisation)**.** *If a sequent is provable in LL, then it has a weakly focalised proof.*

5 Conclusion

We gave a semantic proof of the weak focalisation property. It seems possible to adapt the proof to use $\mathsf{LL}_{\mathrm{Foc}}$ instead of $\mathsf{LL}_{\mathrm{foc}}$ (and thus get the full focalisation property) by considering the submonoid of contexts of the shape $\mathcal{P}, ?\Gamma, \mathcal{X}^\perp$, but I couldn't make it work. Another possible future direction would be to extend this to first-order linear logic.

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